New topographic functionals: RORS and DRS

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Preliminary Draft

This (draft) paper is a continuation of "A new topographic functional" [2], which defines *omnidirectional relief and steepness* (ORS). Refer to that paper for background concepts and notation.

1 Reduced ORS (RORS)

It is common to see a list of the highest N peaks in a region, for example, the top 53[1] peaks in Colorado, known as the "fourteeners"—the peaks exceeding 14,000 feet. However, such height-based lists must include some sort of cutoff criterion, to avoid listing trivial subpeaks (or, in the logical extreme, an infinity of points surrounding the summit of the highest peak on the list). Some measures, notably topographic prominence[1], need no such cutoff; such a measure automatically factors in the independence of a summit, meaning that no trivial subpeak will get a high value. We created RORS to have this feature: it is a measure of a summit's[2] *independent* impressiveness. In particular, we will see that its most important property is that it is *automatically discrete*: for any $\varepsilon > 0$, the set of points $\mathbf{p}$ with $\mathrm{RORS}(\mathbf{p}) > \varepsilon$ is discrete (and hence finite, in a bounded domain). However, our particular definition of RORS involves more choices than we made for ORS, some of which are justified more on aesthetic than mathematical grounds.

To fix ideas, consider the example of the Teton Range in Wyoming. The highest point, and the point with the highest ORS value (ORS = 683 m), is the summit of the Grand Teton. See the topographic map or better, view the range in Google Earth (go to N43.74 W110.8).

If we were to make a list of the "best" points, as judged by ORS, in the range, the summit of the Grand Teton would clearly top the list. But what should be number two? Certainly not the second-highest boulder on the same summit, and perhaps not even nearby peaks such as Mount Owen (just north of the Grand Teton), which is overshadowed significantly by its neighbor, and which could reasonably be considered a subsidiary point on the same massif. The RORS value of Mount Owen will be substantially reduced, compared to its ORS value, by the presence of the Grand Teton nearby. One way to say this is that, given that the Grand Teton has a high ORS value, the fact that Mount Owen has a high ORS value does not convey that much new information, since Mount Owen is part of the same massif. The RORS value of Mount Owen is supposed to reflect, roughly, the relief and steepness that it has *apart* from its being a part of the Grand Teton massif.

[1] The exact number depends on exactly what list is used.

[2] Actually, RORS, like ORS, can be applied to any point on a landscape. However the points with large RORS values tend to be (but are not always) summits.

The number two on the RORS-ranked list for the Teton Range is in fact Mount Moran, which is significantly more independent that Mount Owen. We present ORS and RORS numbers for selected points in the Teton Range in Section 2.

We first define the RORS of a reference point $\mathbf{p}_0$ relative to a specific set of points $\mathbf{p}_1, ..., \mathbf{p}_n$, and a landscape function h.[3] One should think of $\mathbf{p}_0$ as a summit to be evaluated, and $\mathbf{p}_1, ..., \mathbf{p}_n$ as nearby, more impressive summits. To obtain RORS, we modify the integrand in the definition of ORS so that each sample point contributes only to the extent that "viewing" $\mathbf{p}_0$ from $\mathbf{x}$ is "more impressive" than viewing $\mathbf{p}_1, ..., \mathbf{p}_n$. Precisely, we set

$$u_i(\mathbf{x}) = \frac{h(\mathbf{p}_i) - h(\mathbf{x})}{\|\mathbf{p}_i - \mathbf{x}\|} \qquad (i = 0, ..., n)$$

and

$$v_i(\mathbf{x}) = f(u_i(\mathbf{x}))$$

where

$$f(u) = \left(\frac{4}{\pi^3}\left(2u \arctan u - \ln\left(u^2+1\right) - \arctan^2 u\right)\right)^{1/2}$$

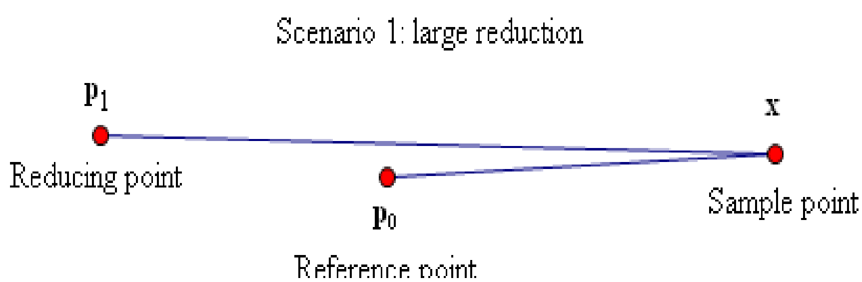
is the modified slope function used in ORS. Then for each $i = 1, ..., n$, $v_0(\mathbf{x}) - v_i(\mathbf{x})$ is a measure of the "impressiveness" of the reference point $\mathbf{p}_0$ as seen from sample point $\mathbf{x}$, masked, or reduced, by the impressiveness of the point $\mathbf{p}_i$. Hence a simple candidate for the new integrand is

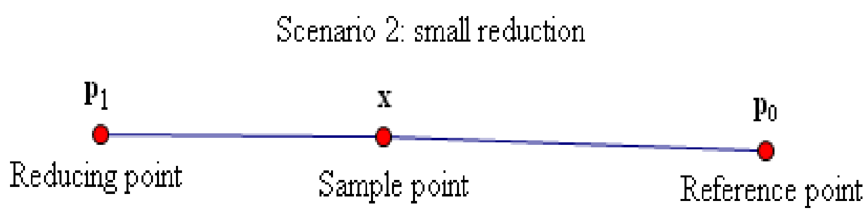
$$\min\left\{\max\left(v_0(\mathbf{x}) - v_i(\mathbf{x}), 0\right)\right\} : i = 1, ..., n$$

Note that taking the max with zero prevents negative contributions; once a nearby peak has stolen all of a certain sample point's contribution to RORS($\mathbf{p}_0$), it can't do any more damage. Similarly, using min (instead of, for example, subtracting the sum of the v_i) lets only the most significant detractor act at each sample point. These are choices we make on empirical and practical grounds; one could use other conventions.

We actually perform one more modification on the functions v_i before collecting them to build the RORS integrand. To explain this, consider two scenarios. In the first, $\mathbf{p}_0$ lies directly between the sample point $\mathbf{x}$ and a better peak $\mathbf{p}_1$; one can think, for example, of $\mathbf{p}_0$ as a subpeak on a ridge of $\mathbf{p}_1$, with the sample point at the base of the ridge. In the second scenario, $\mathbf{p}_0$ and $\mathbf{p}_1$ are diametrically opposed as viewed from $\mathbf{x}$; for example, they could be on opposite sides of a valley, with the sample point on the valley floor. In the latter scenario, it is plausible to consider $\mathbf{p}_0$ as more independent of $\mathbf{p}_1$ than it is in the former, due to the relative position of the two peaks as viewed from the sample point. You can see these two scenarios in the Swiss Alps in Google Earth: Scenario 1, Scenario 2. To distinguish these situations, we introduce an angle weighting, as

[3] In the remainder of the paper we will always use $h(\mathbf{p})$ as the reference height for a reference point $\mathbf{p}$.





follows. For $i = 1, ..., n$, we let $\alpha_i(\mathbf{x})$ be the angle between the rays $\overrightarrow{\mathbf{x}\mathbf{p}_0}$ and $\overrightarrow{\mathbf{x}\mathbf{p}_i}$, and we let

$$w_i(\mathbf{x}) = \frac{1}{2}\left(1 + \cos \alpha_i\left(\mathbf{x}\right)\right) = \cos^2\left(\frac{\alpha_i\left(\mathbf{x}\right)}{2}\right)$$

Note that w_i varies from 1, in the ridge scenario, down to 0, in the valley scenario. We then define the RORS integrand g to be

$$g(\mathbf{x}) = \min\left\{\max(v_0(\mathbf{x}) - w_i(\mathbf{x})v_i(\mathbf{x}), 0) : i = 1, ..., n\right\}$$

and we define[4]

$$\begin{aligned} \mathrm{RORS}(\mathbf{p}_0; \mathbf{p}_1, ..., \mathbf{p}_n; h) &= \|g\|_2 \\ &= \left[\int_{\mathbb{R}^2} g(\mathbf{x})\ dA(\mathbf{x})\right]^{1/2} \end{aligned}$$

This gives a notion of the "impressiveness" of a point as reduced by a specific list of other points. To make a "best" list for a region, one then follows the following procedure to obtain an absolute (not relative) version of RORS. The first point on the list, say $\mathbf{p}_1$ is the maximum of ORS for the region. The second point is the point whose RORS, relative to $\mathbf{p}_1$, is maximum. The third is the

[4] We use single integral signs throughout this paper, in contrast to our use of double integrals for subsets of $\mathbb{R}^2$ in the previous paper. This is to avoid cumbersome quadruple integral notation for DRS.

point whose RORS, relative to $\mathbf{p}_0, \mathbf{p}_1$, is maximum, and so on. This yields a list with the property that the nth entry is the best among all points considered relative to the points above it on the list.

However, if done by the letter, this procedure is obviously cumbersome, especially if we want a long list of best peaks. However, it is easy to make approximations and simplifications that reduce the time required to compute RORS significantly. First, since RORS $\leq$ ORS, one need not consider points that do not have a relatively high ORS value. Second, since the effect of reduction falls off relatively quickly with distance, one need not include far-away peaks as potential reducers. Third, reducing by many points almost never produces much more reduction than reducing by the most "powerful" (usually the closest) two or three reducing points.

Nonetheless, calculating the top 50 points by RORS in a U.S. state, for example, is a compute-intensive process. It is also somewhat sensitive to small errors in the data, but that is unavoidable for a measure of this type—RORS is a "winner-take-all" measure, where two peaks that are close to being tied (and close physically to each other) can get forcibly separated on the list, with one being declared the winner, and the other getting drastically reduced by the winner. It is easy to show that any measure that is automatically discrete will have this property, so this type of sensitivity is unavoidable. (Recall that ORS, on the other hand, is continuous in the input data, and in the cases of interest to us, even Lipschitz. But it is certainly not automatically discrete—it serves a different purpose from RORS.)

2 Examples of RORS calculations

We will present a number of examples of RORS calculations in a later draft of this paper. At this point we refer the reader to our lists on the Peaklist website.

3 Domain Relief and Steepness (DRS)

First we recall from [2] the definition of DRS of a region. Roughly, it is the RMS average of the ORS value for every point in the domain. But note two modifications: first, given a bounded domain $K \subset \mathbb{R}^2$, and a landscape function h, we redefine ORS to use sample points only within the given domain. Second, instead of declaring our modified slope integrand f to have $f(u) = 0$ for $u < 0$, we extend it as an even function.[5]

[5] This change is not essential, but it does make the resulting formula more symmetric. It is easy to verify that using the original convention for f instead results in a definition of DRS that is $1/\sqrt{2}$ times that given here.

Hence, with notation as in Section **??** of [2], we define the new version of ORS, appropriate to this setting, as

$$\begin{aligned} \mathrm{ORS}(\mathbf{p}, h_0; h, K) &= \|f \circ u\|_{2,K} \\ &= \left[\int\limits_K f^2 \left(\frac{h_0 - h(\mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|} \right) \, dA(\mathbf{x}) \right]^{1/2} \end{aligned}$$

and we define

$$\mathrm{DRS}(h, K) = \left[\frac{1}{A(K)} \int\limits_K \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h, K) \, dA(\mathbf{p}) \right]^{1/2}$$

where $A(K)$ is the area of K. (If $A(K) = 0$ we define $\mathrm{DRS}(h, K) = 0$; we will justify this below.) This can be expressed directly in terms of the (new) modified slope integrand f as follows. Abusing notation slightly, let $u(\mathbf{p}, \mathbf{x}) = (h(\mathbf{p}) - h(\mathbf{x})) / \|\mathbf{p} - \mathbf{x}\|$. Then

$$\begin{aligned} \mathrm{DRS}(h, K) &= \frac{1}{\sqrt{A(K)}} \|f \circ u\|_{2,K} \\ &= \left[\frac{1}{A(K)} \int\limits_{K \times K} f^2 \left(\frac{h(\mathbf{p}) - h(\mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|} \right) \, dA(\mathbf{p}) \, dA(\mathbf{x}) \right]^{1/2} \end{aligned}$$

Note that this (quadruple) integral is symmetric in the variables $\mathbf{p}$ and $\mathbf{x}$, and that it has units of length, just as ORS does (recall that f is dimensionless).

Now we turn to results that go further than what we had in [2], but are not yet optimal.

First, we note that DRS satisfies obvious scaling and invariance properties akin to those satisfied by ORS. We won't write them down explicitly. Next, we want to give a simple property of DRS which clarifies exactly how much it is like, and how much it is unlike, and ordinary RMS average. The difference comes from taking sample points only from the region K. For an ordinary average, the following inequality would be an equality.

Lemma 1 *Let the landscape function h be fixed and suppose $K = K_1 \cup K_2$ with K_1, K_2 disjoint. Then*

$$\mathrm{DRS}^2(h, K) \geq \frac{A(K_1)\mathrm{DRS}^2(h, K_1) + A(K_2)\,\mathrm{DRS}^2(h, K_2)}{A(K)}$$

Proof. Let $g(K) = A(K)\mathrm{DRS}^2(h, K)$. Then

$$\begin{aligned} g(K) &= \int\limits_{K\times K} f^2(u(\mathbf{p},\mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\ &= \int\limits_{K_1\times K_1} f^2(u(\mathbf{p},\mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\ &\quad + \int\limits_{K_2\times K_2} f^2(u(\mathbf{p},\mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\ &\quad +2 \int\limits_{K_1\times K_2} f^2(u(\mathbf{p},\mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\ &\geq g(K_1) + g(K_2) \end{aligned}$$

which is what we wanted to show. ■

We refer to this property as the "superadditivity" of DRS (although more properly it is g which is superadditive).

Proposition 2 *DRS is continuous as a function of h in the L^∞ norm.*

Proof. This is clear since DRS is (the square root of) an integral of ORS^2, which is continuous in L^∞. ■

In fact, DRS is substantially better than this simple proposition indicates, since it averages out the variation in ORS. With mild hypotheses, it is probably Lipschitz in h with respect to the L^1 norm. In other words, a tall but skinny feature will contribute only a small amount to DRS. We have not yet worked out the details, however. But even L^∞ continuity is significant, since DRS is a measure of ruggedness, which would ordinarily be calculated with derivatives.

We can also look at continuity in the region K. We define a metric on the set of bounded measurable regions K by taking the area (Lebesgue measure) of the symmetric difference:

$$d(K, K') = m(K \Delta K')$$

There is another way to write this metric. Let χ_K be the characteristic function of K. This is in L^1 exactly when K has finite area. Then it is easy to see that

$$d(K, K') = \|\chi_K - \chi_{K'}\|_1$$

In other words, taking the characteristic function embeds the set of bounded measurable regions isometrically into L^1.

Proposition 3 *Fix a landscape h. Then DRS(h, K) is continuous as a function of K with respect to the metric d.*

Proof. On the set of regions K with positive area, it is enough to show that the function $g(K) = A(K)\mathrm{DRS}^2(h, K)$ is continuous. Note that in general,

$$d(K, K \cup K') \leq d(K, K') \leq d(K, K \cup K') + d(K', K \cup K')$$

Hence we can assume without loss of generality that $K \subset K'$, and we let $L = K' - K$. We have

$$g(K) = \int_{K \times K} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x})$$

so

$$\begin{aligned}
g(K') - g(K) &= \int_{K' \times K'} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) - \int_{K \times K} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\
&= \int_{L \times K} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) + \int_{K \times L} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\
&\quad + \int_{L \times L} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\
&\leq 2\int_L \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h, K)\ dA(\mathbf{p}) + \int_L \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h, L)\ dA(\mathbf{p}) \\
&\leq 2\int_L \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h)\ dA(\mathbf{p}) + \int_L \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h)\ dA(\mathbf{p}) \\
&\leq 3A(L)\, \|h\|_\infty^2 \\
&= 3d(K, K')\, \|h\|_\infty^2
\end{aligned}$$

using the flagpole bound on ORS. Hence g is actually Lipschitz, and DRS is continuous.

Now we just need to show that as K shrinks to zero area, its DRS value (not just $g(K)$) goes to zero. We have

$$\begin{aligned}
\mathrm{DRS}^2(h, K) &= \frac{1}{A(K)} \int_{K \times K} f^2(u(\mathbf{p}, \mathbf{x}))\ dA(\mathbf{p})\ dA(\mathbf{x}) \\
&\leq \frac{4}{\pi^2 A(K)} \int_{K \times K} |u(\mathbf{p}, \mathbf{x})|\ dA(\mathbf{p})\ dA(\mathbf{x}) \\
&\leq \frac{4}{\pi^2} \sup_{\mathbf{x} \in K} \int_K |u(\mathbf{p}, \mathbf{x})|\ dA(\mathbf{p})
\end{aligned}$$

Now, in polar coordinates centered at $\mathbf{x}$,

$$\begin{aligned} |u(\mathbf{p},\mathbf{x})| &= \frac{|h(\mathbf{p})-h(\mathbf{x})|}{\|\mathbf{p}-\mathbf{x}\|} \\ &= \frac{|h(\mathbf{p})-h(\mathbf{x})|}{r} \\ &\leq 2\frac{\|h\|_\infty}{r} \end{aligned}$$

(a.e.) Hence we need to bound

$$\int\limits_K \frac{1}{r} dA(\mathbf{p})$$

which, for a fixed area $A(K) = k$, is clearly maximized in the case where K is a disc of radius $a = \sqrt{k/\pi}$ centered at $\mathbf{x}$, in which case

$$\begin{aligned} \int\limits_K \frac{1}{r} dA(\mathbf{p}) &= \int_0^{2\pi} \int_0^{\sqrt{k/\pi}} dr\ d\theta \\ &= 2\sqrt{\pi k} \end{aligned}$$

and this is independent of $\mathbf{x}$, so

$$\mathrm{DRS}^2(h,K) \leq \frac{16}{\pi^2} \|h\|_\infty \sqrt{\pi A(K)}$$

which clearly shows continuity as $A(K) \to 0$. ■

We now look at some optimization problems for DRS.

Problem 1: Given a fixed landscape h and a set K_0, find a subset $K \subset K_0$ which maximizes $\mathrm{DRS}(h,K)$.

Problem 2: Given a fixed landscape h, a set K_0, and $k > 0$, find a subset $K \subset K_0$ which maximizes $\mathrm{DRS}(h,K)$ subject to the constraint $A(K) = k$.

Problem 1 is a little more natural than Problem 2, since it lacks the area parameter. Note that the fact that $\mathrm{DRS}(h,K) \to 0$ as $A(K) \to 0$ means that this problem will avoid a simple pathology that found be found in most problems of the form "find the region with the greatest average X"—usually, a search for such a region will simply converge on the maximum of X on the region. The superadditivity of DRS avoids such a pathology—a small region will always have a small DRS simply because it includes very few sample points. However in our numerical calculations for real-world landscapes, the optimal region does tend to be fairly small—not surprisingly, the horizontal scale of the optimal region approaches (in order of magnitude, at least) the vertical scale. So it can be a single massif or a small, particularly rugged subrange of a larger range. We will discuss particular examples below.

However, beyond this simple, avoided pathology, there is a much larger problem of whether a minimizing region exists at all, even in Problem 2, with a fixed area. It is unclear whether, without further assumptions on K or h, we will get a sequence of progressively better regions which has no limit (in an appropriate topology). This is of course a classic situation in the calculus of variations, and we have not yet investigated this problem thoroughly. We can lay out a modified problem:

Problem 3: Given a fixed landcape h, in some class $\mathcal{C} \subset L^\infty$, a class $\mathcal{K}$ of allowed regions, and a set $K_0 \in \mathcal{K}$, find a subset $K \subset K_0$, with $K \in \mathcal{K}$, which maximizes $\mathrm{DRS}\,(h, K)$.

We hope that the classes $\mathcal{C}$ and $\mathcal{K}$ need not be too restrictive to guarantee a solution. Two examples of our vague thinking along these lines:

Questions: (1) If h is smooth (say C^1) then can we guarantee a solution to Problem 3, with no a priori restriction on K? Will the optimal K have a relatively nice boundary? Must we make explicit assumptions about the niceness of the set of critical points of h? (2) For an arbitrary $h \in L^\infty$, if we require K to be convex, can we guarantee a solution to Problem 3?

Note that even if one or both of these questions has a positive answer, neither is particularly satisfactory, since both restrictions are rather severe for our setting. Mountain ranges have vertical cliffs, so h is typically not even continuous (although it usually isn't a horribly discontinuous function, so perhaps some sort of piecewise smoothness is an appropriate assumption). And the shape one would expect to get "naturally" (without a priori restriction on K) for a maximizer would not usually be convex (picture the contours of a mountain range). But both questions are reasonable starting points, about which we have though a bit—but we're not yet willing to write anything down.

However, turning from the pure approach to a more applied, numerical approach, we see no signs of any major practical obstacle to solving (approximately) Problems 1 and 2. Coarse-gridded numerical approximations to this problem yield stable results. For example, our calculations indicate that the most rugged region in the contiguous 48 states is the Picket Range of the North Cascades, in Washington State.[3] (And no, the optimal regions don't tend to be convex, or even always connected.)

We won't go into the details of the calculations here (at least for this draft) but we will mention one practical note about how we actually proceed with Problem 2. We actually consider a slightly more general form of DRS, namely

$$\mathrm{DRS}_q(h, K) = \left[\frac{1}{A(K)^q} \int_K \mathrm{ORS}^2(\mathbf{p}, h(\mathbf{p}); h, K)\ dA(\mathbf{p}) \right]^{1/2}$$

Note that the ordinary case is when $q = 1$, and if $q = 0$ then we get the "total" L^2 norm, instead of the "average". So clearly the analog of Problem 1 is silly for the case $q = 0$, as the optimal region will always be all of K_0. But for $0 < q < 1$, the analog of Problem 1 is just as well-defined as it is for $q = 1$, and it will tend to give larger and larger optimal regions as q decreases. It is not much harder to see that adjusting q gives an alternate parametrization to using $A(K)$ for problem 2. This has proved convenient, as it avoids having to deal with the fixed-area constraint in that problem. So in our calculations presented on the peaklist.org website, we have actually looked for overall maximizers of DRS_q for various q, to indirectly solve Problem 2.

It may also very well be the case that some DRS_q with $q \neq 1$ is of as much or more interest in its own right than $\mathrm{DRS} = \mathrm{DRS}_1$. It has an extra arbitrary parameter, and we see no clear reason to pick some paricular $q \neq 1$, which is why we prefer DRS_1. But further investigation may make us prefer some other choice of q.

References

[1] "Topographic prominence", http:en.wikipedia.org/wiki/Topographic_prominence

[2] Edward Earl and David Metzler, "A new topographic functional", submitted to the *American Mathematical Monthly*. Available at http://www.peaklist.org/spire/theory/paper.pdf

[3] http://www.peaklist.org/spire/